Z. phys. Chemie, Leipzig **264** (1983) 3, S. 449–463

Computer and Automation Institute, Hungarian Academy of Sciences, Budapest

Several Exact Results on Deterministic Exotic Kinetics

By **A. Császár**[1], **P. Érdi**[2], **L. Jicsinszky**[1], **J. Tóth**[3] and **T. Turányi**[1]

With 8 Figures and 2 Tables (Received 15th October 1981)

Abstract

Exotic phenomena in the deterministic model of complex chemical reactions are studied on the basis of preliminarily reported results. It is shown that the absence of a special kind of autocatalysis, autoinhibition and cooperativity implies the existence of a unique, asymptotically stable, positive equilibrium point. The class of chemical reactions with gradient system as its deterministic model is delineated. A procedure is given for the construction of oscillatory reactions. A neurobiological application of one of the constructed models is shown.

1. Introduction

Traditional reaction kinetics has been dealing with the large class of chemical reactions characterized by having a unique and stable stationary point. The complementary class of reactions is characterized either by the existence of more than one stationary points, or by an unstable stationary point (which could possibly bifurcate to periodic solutions). Other "extraordinaxities", e.g. Lorenzian type chaotic effects and other chemical strange attractors may occur in this second class. The term *exotic kinetics* refers to different types of qualitative behaviour: to sustained oscillation, instability, multistationarity, and chaotic effects. (Other irregular effects can be expected in higher dimensions.)

Investigation of chemical exotic systems has been executed by *pure* and *applied* sciences as well. Occurence of exotic phenomena in industrial chemical reactors [1–3] and in chemical [4, 5] and biological [5] systems form the experimental basis of the theory.

Exotic chemical phenomena, especially reactions exhibiting multistationarity, can be treated by different multidisciplinary theories: by *catastrophe theory* [6], by the theory of *dissipative structures* [5], and by *synergetics* [7]. Since chemical reaction kinetics is a relatively well-formulated discipline, belonging to the theories of population systems, this theory—sometimes called *formal reaction kinetics*—can be applied as a metalanguage to formulate several non-chemical phenomena.

Though both deterministic and stochastic models are known to describe the temporal evolution of chemical reactions phenomenologically, we restrict ourselves in this paper to the continuous time, continuous state deterministic model.

---

[1] A. Császár, L. Jicsinszky and T. Turányi, Eötvös Loránd University, Budapest and

[2] P. Érdi, 1st Department of Anatomy, Semmelweis University of Medicine, Budapest, and

[3] J. Tóth, Computer and Automation Institute, Hungarian Academy of Sciences, Budapest, Kende u. 13–17., Hungary, H-1111.

Two different approaches will be made here. In the general M-dimensional case we give negative criteria *excluding* exotic behaviour. In the two-dimensional case, however, we present a method of *constructing* kinetic systems with exotic behaviour: with periodic solution. These mean in particular that interdependence between the qualitative behaviour of the reaction system and its stoichiometry will be studied (Section 2). A special conclusion will be given: Some (nonlinear) chemical systems without such—precisely definable—effects as *autocatalysis, autoinihibition* and *cooperativity* are shown to have a unique asymptotically stable positive equilibrium point, if the reaction has a special algebraic structure (i.e. it is *weakly reversible* [8]).

A broad class of differential equations examined by catastrophe theory is the class of gradient systems. The question to be answered here is: Is this kind of differential equations relevant to reaction kinetics? It will be shown that a narrow—and not too important—subclass of chemical reactions can only have a gradient system as its deterministic model (Section 3).

A procedure will be given for the construction of a class of *two-component* chemical systems which can exhibit oscillatory behaviour. The periodic behaviour of several concrete systems has been supported by numerical integration of the kinetic differential equations (Section 4).

The idea of the procedure is used to the construction of a four-variable system. This example—shown in Section 5—has a *neurobiological application* as well.

The present paper is based upon the preliminary reports [18, 24, 29, 30].

2. Structural Condition of Exotic Behaviour in Complex Chemical Systems

2.1. Exotic Chemical Systems: Some Preliminaries

The differential equations describing chemical systems do not generally lend themselves to analytic solution. Linear stability analysis and numerical simulation were the methods most often used to investigate the qualitative properties and the temporal evolution of systems. Another approach derives from Horn and Jackson [8] who established the existence of a Lyapunov function for a broad and easily identifiable class of systems. Subsequently, Horn [9] and Feinberg [10, 11] established criteria relating the qualitative behaviour of the reaction system to the algebraic structure of the underlying reaction mechanism. They gave a *negative criterion* for the existence of exotic behaviour. Here we summarize some of their notations and results.

Let the chemical components or species of the mechanism be $A(1), \dots, A(M)$, the *complexes*

$$C(n) = \sum_{m=1}^{M} y^m(n)\, A(m) \quad (n = 1, 2, \dots N).$$

Therefore the *complex vectors* of stoichiometric coefficients are

$$y(n) = (y^1(n), \dots, y^M(n))^T \quad (n = 1, 2, \dots N).$$

The *elementary reactions* of the mechanism are

$$C(i) \to C(j) \quad (i \neq j; i, j = 1, 2, \dots N).$$

Let L be the number of the connected subgraphs of the directed graph formed by the complexed as vertices and reactions as edges, i.e. L is the number of the *linkage classes.*

The reaction is *weakly reversible,* if the transitive closure of the relation determined by the above defined directed graph (the FHJ graph) is a symmetric relation.

Let s be the dimension of the *stoichiometric space* S, where

$$S := span\{y(j) - y(i);\quad C(i) \to C(j)\}.$$

The *deficiency* of the mechanism is

$$\delta := N - L - s.$$

According to one of the assertions of the zero deficiency theorem [11] if a chemical system with $\delta = 0$ is weakly reversible, then for mass action kinetics with any choice of positive rate constants the existence, uniqueness and asymptotic stability of a positive equilibrium point follow, i.e. the exotic behaviour of these systems is excluded.

The *Volpert-graph* of a chemical mechanism is a directed bipartite graph with multiple edges constructed as follows. Each elementary reaction and each chemical component is represented by a vertex from the two classes of vertices, respectively. The number of directed edges running from $A(m)$ to the vertex representing the elementary reaction $C(i) \to C(j)$ is $y^m(i)$ and the number of directed edges from the vertex representing this reaction to $A(k)$ is $y^k(j)$.

Based upon a theorem of VOLPERT [12, 13] the set of reactions which can exhibit exotic phenomena, can be further narrowed, namely:

If the Volpert-graph of the reaction is acyclic, the existence, uniqueness and asymptotic stability of a nonnegative equilibrium point follow.

Let us turn to studying two-dimensional systems. *Linear stability analysis* gives a necessary condition for limit cycle behaviour of these systems. If A is the matrix of coefficients of the linearized kinetic equation, and $\operatorname{tr} A$ and $\det A$ denote the trace and the determinant of the matrix A, then $\operatorname{tr} A > 0$ and $(\operatorname{tr} A)^2 - 4 \det A < 0$ is a necessary and sufficient condition of the existence of an unstable focus, that may bifurcate to a limit cycle [14].

The *Hanusse-Tyson-Light theorem* [15, 16] is a straight consequence of this condition applied to systems with mass action kinetics: A complex chemical reaction involving two intermediates must have at least one three-molecular elementary reaction in order to show limit cycle behaviour.

A useful idea by FEISTEL and EBELING [17] is that they specialized

$$\dot{x}(t) = F(x(t), (y(t)),$$

$$\dot{y}(t) = G(x(t), y(t))$$

in order to obtain a system of differential equations being able to lead to limit cycle. The specialization has been motivated by subsequent linear stability analysis. Their system of differential equations is:

$$\dot{x}(t) = y(t)f(x(t)) - b(x(t)) - x(t), \tag{1}$$

$$\dot{y}(t) = 1 - y(t)f(x(t)) + b(x(t)). \tag{2}$$

Here a generalization of (1)–(2) will be given, and as a consequence, a class of chemical reactions being possibly candidates of oscillatory behaviour will be delineated.

2.2. Independent Complex Vectors: Non-Exotic Behaviour

Returning to the general M-dimensional case a class of *nonlinear* complex chemical reactions will be excluded from the possible representatives of exotic chemical systems. Technically, a subclass of the class excluded by the zero deficiency theorem will be characterized [18].

The main point is that the deficiency of a chemical mechanism *with independent complex vectors*—except possibly the zero vector corresponding to the empty complex—is zero.

At first, let us suppose that $L = 1$. Then s equals to the number of edges of a spanning tree of the FHJ-graph, i.e. $s = N - 1$. Therefore $\delta = 0$.

In the general case let the number of complexes being contained in the lth linkage class ($l = 1, 2, \ldots, L$) be N_l, then the stoichiometric subspace S is the direct sum of subspaces $S_1, \ldots, S_L$, where the numbering may be chosen so that $\dim S_l = N_l - 1$. Therefore $N = \sum_{l=1}^{L} N_l$, $S_l = \sum_{l=1}^{L} \dim S_l$, thus $\delta = 0$.

Now the zero deficiency theorem can be applied: In a chemical mechanism with independent complex vectors—except possibly the zero vector—

(i) if the mechanism is *not* weakly reversible, then no positive equilibrium concentration exists (for arbitrary, i.e. for not necessarily mass action kinetics);

(ii) if the mechanism *is* weakly reversible, then for mass action kinetics with any choice of positive reaction rate constants, there exists a unique positive equilibrium concentration, every positive equilibrium concentration is asymptotically stable and the kinetic differential equation cannot give rise to nontrivial sustained periodic solutions with positive coordinates.

A special case where independence of the complex vectors is fulfilled will be obtained if all of the components occur in exactly one complex. In this case none of the components occurs as reactant and product in the same reaction:

$$\sum_{m=1}^{M} y^m(i)\, y^m(j) = 0$$

what is equivalent to the exclusion of this special kind of *autocatalysis* and *autoinhibition*. (Pairwise orthogonality of the complex vectors—not necessarily belonging to the same elementary reaction—ensures their independency.)

A further specialization provides the *generalized compartment system*, i.e. the system where it is also true that all of the complexes contain not more than one component. This means that only complexes of the form $y(m)\,A(m)$ ($m = 1, 2, \ldots, M$) are allowed. (*Compartment systems*—often applied in biology—are obtained when all of the stoichiometric coefficients are less than or equal to 1 [19]). We may say that a certain kind of cooperativity is excluded in generalized compartment systems, because no (reactant) complex contains more than one kind of the different chemical components. In other words, this kind of nonlinearity excludes exotic behaviour.

In the case of generalized compartment systems, exotic behaviour can be excluded also for some further, not weakly reversible mechanisms using the results of VOLPERT [13]. In every generalized compartment system having an acyclic FHJ-graph and being endowed with arbitrary kinetics and not containing the zero complex

(i) the solution of the kinetic differential equation is defined and bounded on $[0, +\infty)$;

(ii) the kinetic differential equations cannot give rise to nontrivial sustained periodic solutions with nonnegative coordinates;

(iii) there exists a unique nonnegative equilibrium concentration having at least one zero coordinate;

(iv) there exists a positive constant K such that

$$\int_0^\infty |\dot{c}_m| < K \quad (m = 1, 2, \ldots, M).$$

(Here $c_m(t)$ is the concentration of the mth component in time t.)

The statements of the theorem follow from the fact that the FHJ-graph of a generalized compartment system is acyclic if and only if its Volpert-graph is acyclic. Exotic behaviour in the generalized compartment system

$$2X \rightleftharpoons 3Y \rightarrow 4Z$$

is *not* excluded by our statements, only the first part of the zero deficiency theorem can be applied.

3. Relationships between Catastrophe Theory and Reaction Kinetics

In the qualitative theory of differential equations, especially in catastrophe theory it has turned out that differential equations of the gradient type are relatively easy to deal with [6, 28]. Is this kind of differential equations relevant to reaction kinetics? How is it related to genuine kinetic properties? The special importance of the question is emphasized by the fact that no consensus exists about the logical status of gradient systems in nonequilibrium chemical thermodynamics. Though it is absolutely clear that gradient systems satisfy a symmetry relation similar to the Onsager-relation [7] and the dynamics of a dissipative system can only be derived from a potential in some exceptional situations [5, 20], some effort was made to set up a generalized dissipation potential [20–22].

As even the significance of thermodynamic coupling in chemical reactions has been questioned [23], the flux-force formulation does not seem to be very useful for the case of chemical reactions. What we do here is only to analyze kinetic gradient systems on the base of [24].

The differential equation

$$\dot{c}(t) = f(c(t)), \qquad f \in C^1(R^M, R^M)$$

is *of the gradient type* if there exists a function $V \in C^2(R^M, R^M)$ (called *potential*) such that $f = \operatorname{grad} V$.

A complex chemical reaction (or mechanism) is a gradient system, if its kinetic differential equation is of the gradient type.

A special class of reactions will be defined: A mechanism is *cross-catalytic* if for each elementary reaction $C(j) \to C(i)$ $(i, j = 1, 2, \dots, N;\ i \neq j)$ either

(i) the coordinates of the *elementary reaction vector* $x(i, j) := y(i) - y(j)$ are non-negative (thus, there is at least one positive among them), or

(ii) the reactant complex vector is a multiple of one of the base vectors of R^M.

In other words, in a cross-catalytic mechanism there does not exist any chemical component causing the decrease of another. Nothing has been claimed about the effect of components on themselves.

The reaction $X + Y \to 2Y$ is not cross-catalytic, since the elementary reaction vector is $(-1, 1)^T$, and the reactant complex vector $(1, 1)^T$ is not a multiple of a base vector. The reaction $X + Y \to 2X + 2Y$ is cross-catalytic, although this reaction does not obey the law of material balance. The notion of *conservativity*, a generalization of obeying the law of atomic balance introduced by HORN and JACKSON [8], will be useful:

A mechanism is *conservative* if there exists a vector $r \in R^M$ with positive coordinates, orthogonal to all of the elementary reaction vectors (or to the stoichiometric space).

Obviously, if a cross-catalytic mechanism contains elementary reactions of type (i) too, then it is not conservative.

Realistic chemical mechanisms consist of reactant complexes not longer than two. A mechanism is called *weakly realistic* if its reactant complexes $C(i)$ $(i \in \{1, 2, \dots, N\})$ belong to one of the following two types:

(i) there exists an $m \in \{1, 2, \dots, M\}$ such that $C(i) = y^m(i)\, A(m)$;

(ii) $C(i) = \sum\limits_{m=1}^{M} y^m(i)\, A(m)$; $y^m(i) = 0$ or 1 for all $m \in \{1, 2, \dots, M\}$.

Realistic chemical systems and generalized compartment systems are weakly realistic.

According to a theorem [24] a weakly realistic mechanism of the gradient type is cross-catalytic. Let us suppose, on the contrary, that there is a term $-kc_m c_{m'} P$ on the right side of the mth equation of the kinetic differential equation; here $k \in R^+$, $m, m' \in \{1, 2, \dots, M\}$, $m \neq m'$ and P is a—possibly void—product of certain $c_i - s$ $(i \neq m, m')$. A well-known necessary and sufficient condition for the existence of a potential for the equation $\dot{x} = f \circ x$, where $f = (f_1, \dots, f_M)$, is: $\partial_i f_j = \partial_j f_i$ $(i, j \in \{1, 2, \dots, M\})$ ([25], 10·35 Remarks). In our case

$$\partial_{m'} f_m = kc_m P + \dots = \partial_m f_{m'}$$

should hold, i.e. $f_{m'}$ should contain the term $-kc_m^2 P/2$. This is impossible, because this term has a negative coefficient and it does not contain $c_{m'}$ [26].

As a corollary of the theorem a necessary condition for a system to be a conservative gradient system can be given: Conservative gradient systems may only be found in two classes of mechanisms:

(α) in the class of non-weakly realistic and non-cross-catalytic ones, and

(β) in the class of those cross-catalytic ones that only contain elementary reactions of type (ii).

A closed compartment system is an element of the class (β). In the special case when the matrix of reaction rate constants is symmetric, the system is of the gradient type.

Two remarks are to be added to the connection between kinetic and thermodynamic gradient systems:

(i) The usual procedure is to transform the kinetic equation $\dot{c}(t) = f(c(t))$ by $y(t) := \varphi(c(t))$ so as to obtain $f(c(t)) = V'(y(t))$, where $V \in C^2(R^M, R)$. Naturally, the thermodynamic equation $\dot{c} = V'oy$ is not a gradient system. It would help to clarify the obscure situation if we could determine when will a gradient system be obtained by a transformation of the above type?

(ii) Motivated by the fact that sometimes a generalized dissipation potential V can be constructed by decomposing a thermodynamic flux into a sum of two fluxes:

$$J = \operatorname{grad} V + u$$

where $V \in \varrho^2(R^M, R)$ and $u: R^M \to R^M$ such that $\alpha^T u(x) = 0$ holds for all $x \in R^M$, the notion of a *generalized kinetic gradient system* can be introduced:

A differential equation $\dot{x}(t) = f(x(t))$ is of the *generalized gradient type* if there exists two functions $V \in \varrho^2(R^M, R)$ and $u \in \varrho^1(R^M, R^M)$ such that

$$f = \operatorname{grad} V + u = V' + u,$$

and

$$x^T u(x) = 0 \quad (x \in R^M).$$

This decomposition is clearly non-unique, namely if

$$f = V_1' + u_1 = V_2' + u_2$$

then $V_1 - V_2$ is a homogeneous function of order 0, and $u_2 - u_1$ is a homogeneous function of order -1, and $(V_1 - V_2)' = u_2 - u_1$ holds (and not necessarily $V_1 - V_2 = 0$ and $u_2 - u_1 = 0$).

It is an open question which of the kinetic differential equations are of the generalized gradient type. It would be also interesting what is the chemical (thermodynamical) meaning of the quantity $\psi(t) := x^2(t)/2 - V(x(t))$ being constant even in the case of generalized gradient systems.

4. Candidates of Limit Cycle Behaviour: Two-Component Systems

Feistel and Ebeling [17] considered (1)–(2) as the deterministic model of chemical reactions with two intermediates. They used this system because the coordinates of the equilibrium point are easy to calculate. One of its coordinates is 1, and the other can be determined as a function of the first one:

$$(x^*, y^*) = \left(1, \frac{1 + b(1)}{f(1)}\right)$$

This principle can be used for generalizing (1)–(2), and thus the following system of differential equations is obtained:

$$\dot{x}(t) = g(y(t))f(x(t)) - b(x(t)) + h(y(t)) - x(t)\,c(y(t)), \tag{3}$$

$$\dot{y}(t) = -g(y(t))f(x(t)) + b(x(t)) - h(y(t)) + c(y(t)). \tag{4}$$

Assuming that $c(y^*) \neq 0$ and knowing the functions b, c, f, g, h the second coordinate of the stationary point can be calculated from the relationship

$$h(y^*) - c(y^*) + g(y^*)f(1) = b(1)$$

and

$$x^* = 1, \text{ thus } (x^*, y^*) = (1, y^*).$$

A practical disadvantage of (3)–(4) is that the Volterra-Lotka model and the Brusselator cannot be derived from them as special cases. A simple common generalization of the kinetic differential equation of these celebrated models is:

$$\dot{x}(t) = g(y(t))f(x(t)) - b(x(t)) + l(x(t)), \tag{5}$$

$$\dot{y}(t) = -g(y(t))f(x(t)) + b(x(t)) - h(y(t)). \tag{6}$$

The functions b, f, g, h, l have to be specialized as shown in Table 1.

Now we give a common generalization of (3)–(4) and (5)–(6):

$$\dot{x}(t) = g(y(t))f(x(t)) - b(x(t)) + k(x(t))\,[h(y(t)) - x(t)\,c(y(t)) - l(x(t))] + l(x(t)), \tag{7}$$

$$\dot{y}(t) = -g(y(t))f(x(t)) + b(x(t)) - h(y(t)) + c(y(t))\,k(x(t)). \tag{8}$$

If $k(x) \equiv 1$, we get (3) and (4), if $k(x) \equiv 0$, we get (5) and (6).

Assuming that $c(y^*) \neq 0$ and $k(x^*) \neq 0$, the following relation holds

$$x^* = 1 + \frac{[h(y^*) - l(x^*)]\,[k(x^*) - 1]}{k(x^*)\,c(y^*)}.$$

To simplify this formula to $x^* = 1$ two special assumptions are possible:

(i) $h(y^*) = l(x^*)$, and

(ii) $k(x^*) = 1$.

Table 1. Specification of functions to appear in (5) and (6)

	Volterra-Lotka model	Brusselator
$b(x)$	0	k_2x
$f(x)$	$-k_2x$	k_3x^2
$g(y)$	y	y
$h(y)$	k_3y	0
$l(x)$	k_1x	$k_1 - k_4x$

Table 2. Several examples: candidates to exhibit limit cycle behaviour

	A 1	A 2	A 3
Complex chemical reaction	$2X + Y \to 3X$ $2X \to X + Y$ $Y \to X$ $\nwarrow\ \swarrow$ 0	$2X + Y \to 3X$ $X \rightleftharpoons Y \to 2Y$ $2X \to X + Y \to 0$ $X + Y \to Y$	$2X \rightleftharpoons 3X$ $X + Y \rightleftharpoons 2X$ $X \rightleftharpoons Y$ $\swarrow\!\!\nearrow\ \nearrow$ 0
$b(x)$	$0\cdot1x^2$	$0\cdot4x^2 + 0\cdot15x$	$0\cdot04x^3 + 0\cdot07x^2$
$c(y)$	0 (or: arbitrary)	y	$0\cdot1$
$f(x)$	$0\cdot1x^2 + 0\cdot05$	$0\cdot6x^2 + 0\cdot2x$	$5\cdot5x^2 + x + 1$
$g(y)$	y	y	$0\cdot02y$
$h(y)$	-1	$0\cdot3y$	$-0\cdot93$
$k(x)$	0	1	x
$l(x)$	$-x$	0 (or: arbitrary)	$-0\cdot1x + 0\cdot07$
Kinetic differential equation	$\dot{x} = 0\cdot1x^2y - 0\cdot1x^2 + 0\cdot05y - x$ $\dot{y} = 0\cdot1x^2y + 0\cdot1x^2 - 0\cdot05y + 1$	$\dot{x} = 0\cdot6x^2y - 0.4x^2 - 0\cdot8xy - 0\cdot15x + 0\cdot3y$ $\dot{y} = -0\cdot6x^2y + 0\cdot4x^2 - 0\cdot2xy + 0\cdot15x + 0\cdot7y$	$\dot{x} = 0\cdot11x^2y + 0\cdot02xy + 0\cdot02y - 0\cdot04x^3 - 0\cdot07x^2 - 1\cdot1x + 0\cdot07$ $\dot{y} = -0\cdot11x^2y - 0\cdot02xy - 0\cdot02y + 0\cdot04x^3 + 0\cdot07x^2 + 0\cdot1x + 0\cdot93$
(x^*, y^*)	$(1, 7\cdot3)$	$(1, 5\cdot5)$	$(1, 7\cdot6)$
tr A	$0\cdot117$	$1\cdot155$	$0\cdot314$
$(\text{tr } A)^2 - 4$ det A	$-0\cdot586$	$-0\cdot358$	$-0\cdot501$
Deficiency	2	3	2

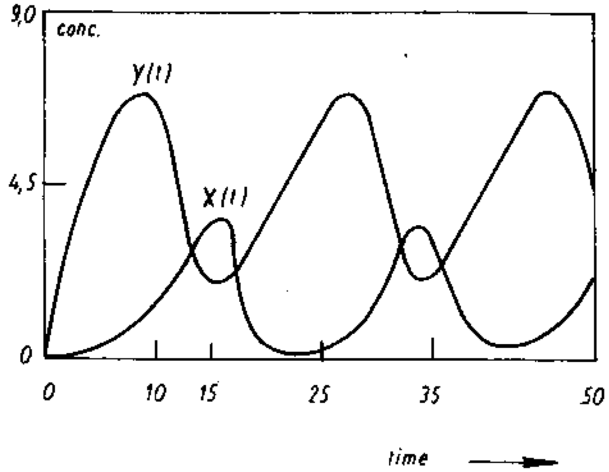


Fig. 1. Time evolution of concentrations in reaction A 1

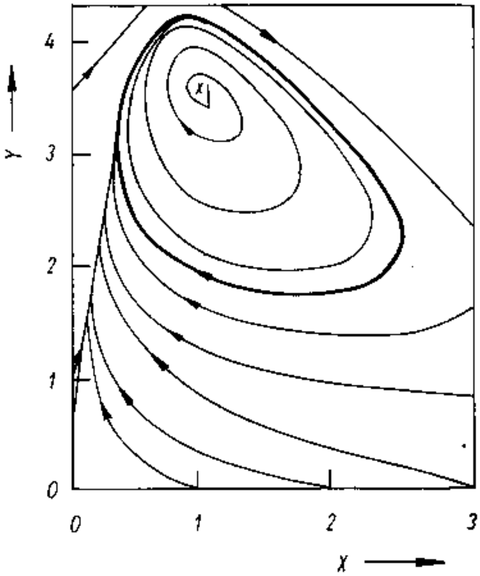


Fig. 2. Selectivity curves of the reaction A 1 with different initial concentrations

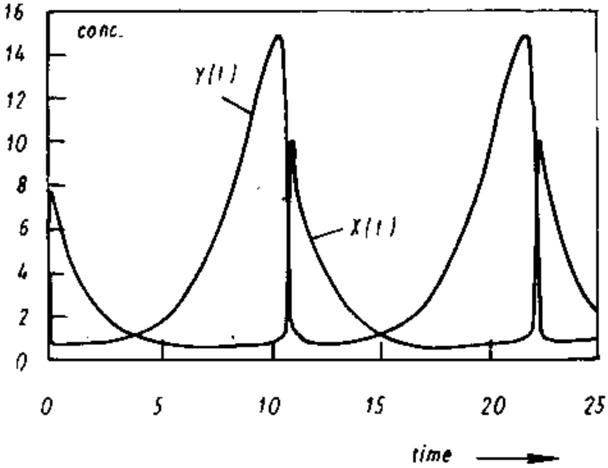


Fig. 3. Time evolution of concentrations in reaction A 2

Reactions having (7)–(8) as their kinetic differential equation may exhibit oscillatory behaviour, especially if their stationary point is an unstable focus. Several examples are shown in Table 2. (Here we have to emphasize that the correspondence between reactions and kinetic differential equations is not unique by far [26].)

The reactions of Table 2 are similar, but different from those presented by SCHNAKENBERG [27].

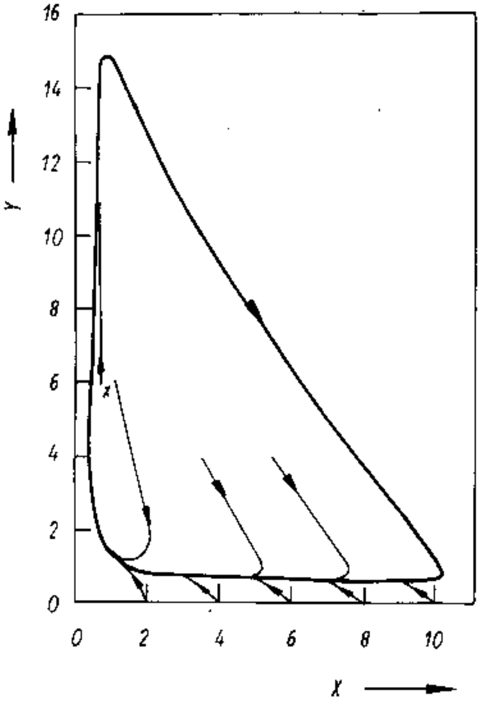


Fig. 4. Selectivity curves of the reaction *A* 2 with different initial concentrations

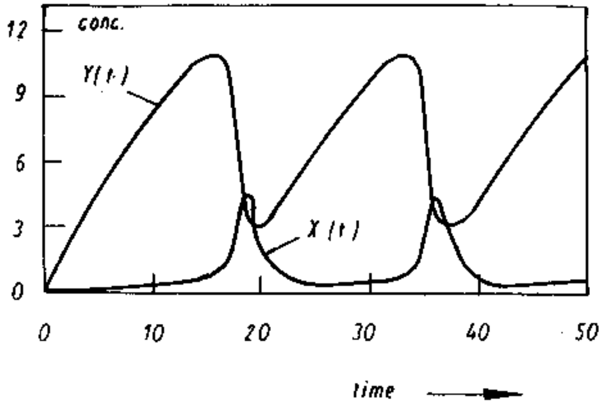


Fig. 5. Time evolution of concentrations in reaction *A* 3

The necessary conditions for the limit cycle behaviour are fulfilled and the numerical integration of the kinetic differential equations support that the reactions show oscillations. The temporal evolution of the concentrations with several initial conditions and the selectivity curves can be seen on Figs. 1–6.

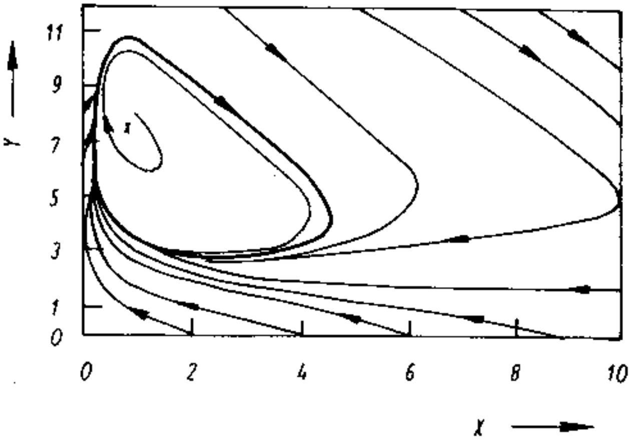
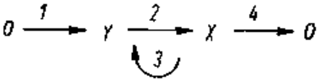


Fig. 6. Selectivity curves of the reaction *A* 3 with different initial concentrations

5. Oscillatory Behaviour of a Four-Component System: A Neurobiological Example

An open chemical system with two intermediates showing oscillatory behaviour is given by the scheme below:



(9)

The processes involved are:

1: inflow,
2: conversion,
3: autocatalysis,
4: outflow.

A four-component closed system can be constructed by slightly modifying the scheme (9):

$$\begin{aligned}&\mathrm{C} \rightarrow \mathrm{Y}\\&\mathrm{Y} \rightarrow \mathrm{X}\\&2\mathrm{X} + \mathrm{Y} \rightarrow 3\mathrm{X}\\&\mathrm{X} \rightarrow \mathrm{B}\\&\mathrm{B} \rightarrow \mathrm{C}\end{aligned} \tag{10}$$

The numerical solution of the kinetic differential equation of (10) can be seen on Fig. 7. This model has oscillatory solutions thus being a possible analogue of the example by Ivanova [13]. As it is known, only (asymptotically stable) limit cycles are impossible in closed systems [28].

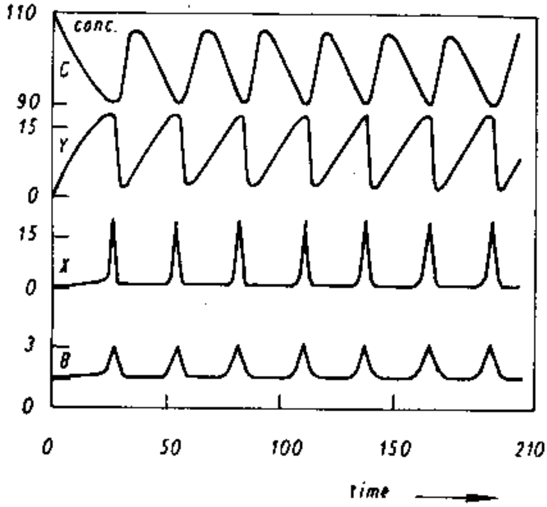


Fig. 7. Time evolution of concentrations in reaction (10). When this reaction is used to model the slow wave, the identifications $X := $ free ACh is the cytosol, and $Y := $ Ch (Cf. Fig. 8 below) are made

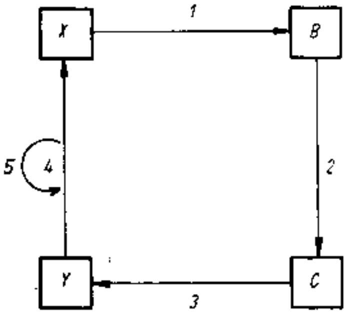


Fig. 8. Skeleton of a lumped model for the explanation of slow wave of ACh. Concentrations: X = cytoplasmic [ACh]; B = [ACh] at the postsynaptic membrane surface; C, Y = [choline]. Subprocesses: 1 = diffusion of ACh, cleft processes; 2 = hydrolysis of ACh; 3 = diffusion of choline; 4 = reuptake of choline; 5 = autocatalytic synthesis of ACh

This reaction is a possible model of the recycling of transmitter during neurochemical transmission. Chemically mediated synaptic transmission is considered to be a dynamic process. The different subprocesses of the transmission (synthesis, storage, release, cleft processes, transmitter-receptor interaction, hydrolysys, reuptake) are treated as interacting ones. At least three different neurochemical and neurophysiological oscillatory phenomena appear at different hierarchical levels of synaptic transmission [29, 30]. The *slow wave* of free cytoplasmic acetylcholine (ACh) were presented by Dunant et al. [31].

This slow wave seems to be the result of integrated activity of the synaptic metabolic subprocesses. The skeleton of a lumped model for the explanation of the slow wave can be seen on Fig. 8.

To simplify the model most of the subprocesses are assumed to be linear. An autocatalytic process (synthesis) is superimposed on the linear uptake process of choline.

The model just described is analogous with reaction (10).

6. Discussion and Outlook

Here we presented several *general* results providing negative criteria for exotic phenomena in the deterministic model of complex chemical reactions. In connection with this was determined two classes of reactions that may contain *gradient systems*. Then we presented a method for constructing oscillatory reactions, and we have shown several examples, one of them having a neurobiological application.

We think the most important trend on the area of exotic chemical phenomena are:

(i) The construction of further general negative criteria, i.e. criteria for the exclusion of exotic behaviour cf. [11];

(ii) stating *general* positive theorems on the existence of exotic phenomena;

(iii) the investigation of these phenomena in the stochastic model and relating the results cf. [32];

(iv) the application of models with exotic characteristics, especially to biological sciences.

References

[1] Aris, R.: Ann. N.Y. Acad. Sci. **316** (1979) 314.
[2] Calo, J. M., and H. C. Chang: Chem. Eng. Sci. **264** (1980) 35.
[3] Luss, D.: In: Dynamics and Modelling of Reactive Systems. Stewart et al. (eds.). New York and London: Academic Press 1980, pp. 131–159.
[4] Tyson, J. J.: Lecture Notes in Biomathematics, Vol. 10, Berlin-Heidelberg-New York: Springer 1976.
[5] Nicolis, G., and I. Prigogine: Self-Organization in Non-equilibrium Systems. New York-London-Sydney-Toronto: Wiley 1977.
[6] Thom, R.: Stabilité Structurelle et Morphogénese. New York: Benjamin 1972.
[7] Haken, H.: Synergetics. Berlin-Heidelberg-New York: Springer 1978.
[8] Horn, F., and R. Jackson: Arch. Rational Mech. Anal. **47** (1972) 81.
[9] Horn, F.: Arch. Rational Mech. Anal. **49** (1972) 172.
[10] Feinberg, M.: Arch. Rational Mech. Anal. **49** (1973) 187.
[11] Feinberg, M.: In: Dynamics and Modelling of Reactive Systems. Stewart et al. (eds.). New York and London: Academic Press 1980, pp. 1–69.
[12] Vol'pert, A. I.: Mat. sbornik **88** (1972) 578.
[13] Vol'pert, A. I., and C. I. Chudyaev: Analiz v klassakh razryvnykh funktsii i uravneniya matematicheskoi fiziki. Moscow: Nauka 1975, p. 363.
[14] Andronov, A. A., A. A. Vitt and C. E. Khaikin: Theory of Oscillators. Oxford: Pergamon 1966.
[15] Hanusse, P.: C. R. Acad. Sci. Ser. C **274** (1972) 1245.
[16] Tyson, J., and J. Light: J. Chem. Phys. **59** (1973) 4164.
[17] Feistel, R., and W. Ebeling: Physica **93A** (1978) 114.
[18] Tóth, J.: React. Kinet. Catal. Lett. **9** (1978) 377.
[19] Jacquez, J. A.: Compartmental Analysis in Biology and Medicine. Amsterdam-London-New York: Elsevir 1972.

[20] Bataille, J., D. G. B. Edelen and J. Kestin: J. Non-Equilib. Thermodyn. **3** (1978) 153.
[21] Edelen, D. G. B.: Int. J. Engng. Sci. **11** (1973) 819.
[22] Gyarmati, I.: Per. Polytechn. **5** (1961) 321.
[23] Keizer, J.: J. Theor. Biol. **49** (1975) 323.
[24] Tóth, J.: React. Kinet. Catal. Lett. **12** (1979) 253.
[25] Rudin, W.: Principles of Mathematical Analysis, 3rd ed. New York: McGraw-Hill 1964.
[26] Hárs, V., and J. Tóth: Colloq. Math. Soc. Janos Bolyai 30 (Qual. Theory of Diff. Eqs. Szeged, 1979) 363.
[27] Schnakenberg, J.: J. Theor. Biol. **81** (1979) 389.
[28] Hirsch, M. W., and S. Smale: Differential Equations Dynamical Systems and Linear Algebra, New York-San Francisco-London: Academic Press 1974, p. 252.
[29] Érdi, P., V. Hárs and J. Tóth in: Proc. Intl. Union Physiol. Sci., Vol. XIV. 1980.
[30] Érdi, P., and J. Tóth: Adv. Physiol. Sci. 34 Math. and Comp. Methods in Physiol., Budapest 1980, S. 113.
[31] Dunant, Y., M. Israel, Lesbats and R. Manaranche: Brain. Res. **125** (1977) 123.
[32] Érdi, P., J. Tóth and V. Hárs: Colloq. Math. Soc. János Bolyai 30. Qual. Theory of Diff. Eqs., Szeged 1979, S. 205.